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APPLICATIONS OF THE TOPOLOGICAL INDEX IN COMPLEX FUNCTION THEORY

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TABLE OF CONTENTS

		Page
ACKNOWLEDGEMENT		ii
INTRODUCTION		1
Chapter		
I.	THE TOPOLOGICAL INDEX	4
II.	APPLICATIONS TO COMPLEX VARIABLES	46
III.	STABILITY	56
IV.	SECOND-ORDER AUTONOMOUS SYSTEMS OF DIFFERENTIAL EQUATIONS	64
BIBLIOGRAPHY		71

INTRODUCTION

The purpose of this study is to present an elementary exposition of the concept of topological index for complex functions and of its applications to several branches of the theory of complex functions. There is also a short illustration of an application of the topological index to second-order autonomous systems of differential equations.

The fundamental concepts needed in the study are presented in the first chapter, where the topological index of a continuous complex function is defined. The basic properties of the index which follow from its definition are also presented in this chapter. Most of the material of this part is a detailed exposition of the topological index as presented by G. T. Whyburn [1]. The index is defined in terms of exponential representations, thus avoiding the use of contour integration. The concept of exponential representation is closely related to that of change of argument. A development of the topological index by means of the concept of change of argument can be found in a book by T. Radó [2]. Of these three methods of defining the topological index, the one using the concept of exponential representations seems to be the simplest. The first chapter ends with the proof that the definition of the topological index of an analytic function agrees with the classical definition, which depends upon the concept of contour integral. The classical definition is indeed the one which is found in most books on complex function theory. The textbook on complex analysis by L. V. Ahlfors [3] contains a detailed development of the topological index of an analytic function with the use of the concept of contour integration.

The second chapter consists of the derivations of certain theorems in classical complex function theory which follow from the properties of the topological index. Most of these theorems deal with the properties of the zeros of analytic functions. A generalized version of Rouché's theorem is presented (theorem A-1). A proof is presented of the fact that an analytic function maps open sets of the complex plane onto open sets of the complex plane. From this result follow several classical theorems: the fundamental theorem of algebra and the maximum modulus principle. These as well as other applications of this type can be found in [1].

The third chapter deals with the concept of topological stability. The definition of the concept of topological stability and an example which illustrates this definition are presented. Certain general results concerning topological stability are proved and the concept of topological equivalence of continuous functions is presented. This chapter ends with several applications of the topological index to the case of a continuous complex function.

The fourth and last chapter deals with second-order autonomous systems of differential equations. The main theorem states a necessary and sufficient condition, in terms of the topological index, that a simple singular point of a second-order autonomous system of differential equations be a saddle point. The proof of this theorem makes use of several theorems about autonomous systems of differential equations which can be found in a book by W. Hurewicz [4].

An alternative method of defining the index is by means of homology and cohomology degree. This method is not as elementary as the one presented here. However, it has the advantage that it can be used in more general situations. Such a development of the topological index is given in [2].

CHAPTER I

THE TOPOLOGICAL INDEX

Let Z denote the complex plane. This chapter will be concerned with functions from certain metric spaces to Z.

A. Admissible exponential representations. Throughout this section let X denote a connected metric space. This section is concerned with a continuous function f from X to Z such that $f(X) \neq Z$.

Definition A-1. Let $p \varepsilon Z-f(X)$. Then $e^{u(x)}$ is said to be an admissible exponential representation of $f(x)-p$ on X if and only if u is a continuous function from X to Z such that

$$e^{u(x)} = f(x)-p \quad \text{for} \quad x \varepsilon X.$$

The function u is said to be a continuous branch of the logarithm of $f(x)-p$ on X.

Theorem A-1. Let $p \varepsilon Z-f(X)$. Suppose that there exists a ray pq with origin at p and contained in $Z-f(X)$. The following three statements are true:

(i) $f(x)-p$ has an admissible exponential representation on X,

(ii) for any continuous branch u of the logarithm of

$f(x)-p$ on X, and for any a, b in X it is true that

$$|\mathrm{Im}(u(a))-\mathrm{Im}(u(b))| < 2\pi, \text{ and}$$

(iii) $u(a)$ may be preassigned any value of $\log(f(a)-p)$.

Proof. Let a be any point of X. This point will remain fixed throughout this argument. Set $u(a)=\log|f(a)-p| + iv$, where v is one of the arguments of $f(a)-p$.

Let the function u be defined by

$$u(x) = \log|f(x)-p| + i(v + v_x) \quad \text{for} \quad x \varepsilon X,$$

where v_x is the signed angle from the ray $pf(a)$ to the ray $pf(x)$ described in such a way that the sector $f(a)pf(x)$ does not intersect the ray pq. It is easily seen that u is continuous on X. This proves (i) and (iii). To prove (ii), suppose that u is a continuous branch of the logarithm of $f(x)-p$ on X. Since the ray with origin at p and direction pq does not intersect $f(X)$, it must be true that $\mathrm{Im}(u(x)) \neq \arg(q-p)$ for any argument of $q-p$ and for $x \varepsilon X$. Let v_o be one value of $\arg(q-p)$. Then $\mathrm{Im}(u(x) \neq v_o + 2n\pi$ for any integer n. Since u is continuous on X, $\mathrm{Im}(u)$ is also continuous on X. The fact that X is connected implies therefore that $(\mathrm{Im}(u))(X)$ is contained in an open interval of length 2π, which proves (ii).

Theorem A-2. If X is a real interval $[a,b]$, a ray, a line, or a plane, and if $p \varepsilon Z-f(X)$, then $f(x)-p$ has an admissible exponential representation $e^{u(x)}$ on X. Moreover, u is uniquely determined up to an additive constant.

Proof. First suppose that $X = [a,b]$, a real, closed interval. Choose a partition $\{x_0, \ldots, x_n\}$ of $[a,b]$ such that $f([x_i, x_{i+1}])$ lies inside a circle not enclosing p, for $i = 0, 1, \ldots, n-1$. This can obviously be done, since $f([a,b])$ is a compact set and $p \varepsilon Z-f([a,b])$.

By Theorem A-1, $f(x)-p$ has an admissible exponential representation $e^{u_i(x)}$ on $[x_i, x_{i+1}]$ for each $i = 0, 1, \ldots, n-1$. Again by theorem A-1, it is possible to set $u_i(x_i) = u_{i+1}(x_i)$ for each $i=1, \ldots, n-1$.

Define a function u on $[a,b]$ as follows:

$$u(x) = u_i(x) \text{ if } x \varepsilon [x_{i-1}, x_i] \text{ for some } i=1, \ldots, n.$$

Each one of the functions u_i is continuous. From this, and from the definition of the function u, it is seen that u is continuous on $[a,b]$. It is clear that

$$e^{u(x)} = f(x)-p \text{ for } x \varepsilon [a,b].$$

Secondly notice that the same procedure can be applied to the case in which X is a ray or a line.

It must be shown that u is unique up to an additive constant. Suppose that $e^{w(x)}$ is another admissible exponential representation of $f(x)-p$ on X, then $e^{u(x)} = e^{w(x)}$ for $x \varepsilon X$. Thus $e^{u(x)-w(x)} = 1$ for $x \varepsilon X$. Since X is connected and $u-w$ is continuous, it follows that

$$u(x) - w(x) = 2\pi k i \text{ for } x \varepsilon X,$$

where k is a fixed integer. The theorem is proved for the case in which X is a real interval $[a,b]$, a ray, or a line.

Now let X be a plane. The plane X will be considered as the complex plane with cartesian coordinates (s,t) so that $x=s + ti$ if $x \varepsilon X$. Consider the real axis $t = 0$. It was shown that $f(x)-p$ has an admissible exponential representation on this line, giving a continuous branch u_0 of $\log(f(x)-p)$. There exists, for each s_o, a continuous branch u_{s_o} of $\log(f(x)-p)$ on the line $s=s_o$. It has been seen that u_{s_o} can be chosen so that $u_{s_o}(s_o,0) = u_o(s_o,0)$. If $x = (s,t) \varepsilon X$, define $u(x) = u_s(s,t)$. It follows that

$$e^{u(x)} = f(x)-p \text{ for } x \varepsilon X .$$

It is necessary to show that u is continuous on X. To do this, the following lemma will be used.

Let R be the rectangle $[s_1, s_2] \times [t_1, t_2]$. Suppose that

(i) $|\arg(f(x_1)-p)-\arg(f(x_2)-p)| < \pi$ for all $x_1 \varepsilon R, x_2 \varepsilon R$,

and

(ii) u is continuous on the base of R, that is, on the set $[s_1, s_2] \times \{t_1\} = B$. The function u is then continuous on R.

Proof of the lemma. By (i), there exists a ray from p which does not intersect $f(R)$. Then, by theorem A-1, there exists a continuous complex function v on R such that

$$e^{v(x)} = f(x)-p \quad \text{for} \quad x \varepsilon R.$$

Again by theorem A-1, it is possible to choose v so that

$$v(s_1, t_1) = u(s_1, t_1).$$

Suppose that $s_1 \leq s \leq s_2$. Let $q = s-s_1 + t_2 - t_1$. Define a function w as follows:

$$w(x) = v(x) - u(x) \quad \text{for } x \varepsilon R.$$

Now define a complex function g on the interval $[0, q]$ by

$$g(r) = w(s_1 + r, t_1) \text{ for } r \varepsilon [0, s-s_1], \text{ and}$$

$$g(r) = w(s, t_1 + r - s + s_1) \text{ for } r \varepsilon (s - s_1, q].$$

It is clear that w is continuous on B, since u and v are continuous on B. The function u is continuous on the vertical line through (s, t_1) and (s, t_2), where u agrees with u_s. Since v is continuous on R, v is continuous on the vertical line segment joining (s, t_1) and (s, t_2). It follows that g is continuous on $[0, q]$.

Since $e^{v(s)} = e^{u(x)} = f(x) - p$ for $x \varepsilon R$, it is true that $e^{u(x) - v(x)} = 1$ for $x \varepsilon R$. Hence $e^{g(r)}$ is identically equal to 1 on $[0, q]$. Then it must be true that $g(r) = 2\pi k i$ for $x \varepsilon [0, q]$ for some fixed integer k, because $[o, q]$ is connected and g is continuous on $[o, q]$.

From the way in which g is defined it is seen that

$$g(0) = w(s_1, t_1) = v(s_1, t_1) - u(s_1, t_1) = 0 ,$$

and therefore

$$g(r) = 0 \text{ for } r \varepsilon [o, q].$$

Since s is an arbitrary number in $[s_1, s_2]$, it follows that

$$w(x) = 0 \quad \text{for } x \varepsilon R.$$

This means that

$$u(x) = v(x) \text{ for } x \varepsilon R.$$

The fact that v is continuous on R implies that u is continuous on R. The proof of the lemma is therefore complete.

Let Q be any rectangle with a base on the s-axis. The set Q is therefore compact. Since f is continuous on Q, f must be uniformly continuous on Q. It is therefore possible to partition Q by means of vertical and horizontal lines into a finite number of subrectangles so that condition (i) of the lemma holds in each of these subrectangles.

Suppose that $x \varepsilon Q$, then z can be joined to the base of Q which is on the s-axis by means of a chain $Q_1, \ldots, Q_n$ of subrectangles of Q such that

(a) Q_i and Q_{i-1} have a base in common for $i=2, \ldots, n$,

(b) $z \varepsilon Q_n$, and

(c) Q_1 has a base on the s-axis.

Applying the lemma repeatedly to $Q_1, \ldots, Q_n$, it follows that u is continuous on Q_n, and hence continuous at z. This shows that u is continuous on Q. Since Q is an arbitrary rectangle having a base on the s-axis, it follows that u is continuous on the whole plane, that is, on X. The uniqueness of u up to an additive constant follows as before, since X is connected. Theorem A-2 is therefore completely proved.

Theorem A-3. If X is a space homeomorphic to any of the spaces described in the hypothesis of theorem A-2, then the conclusions of theorem A-2 hold for X.

Proof. Let Y be any of the spaces described in the hypothesis of theorem A-2, and let h be a homeomorphism from Y on to X. Define v to be the composite function fh. Since h is continuous on Y and f is continuous on X, it follows that v is continuous on Y.

By hypothesis, $v(x)-p$ has an admissible exponential representation $e^{u(x)}$ on Y. Thus

$$e^{u(x)} = v(x) - p \quad \text{for } x \varepsilon Y.$$

(Notice that the equality $v(x)-p = 0$ is impossible if $x \varepsilon Y$.) This means that

$$e^{uh^{-1}}(z) = f(z)-p \text{ for } z \varepsilon X,$$

since h is one-to-one and onto. It is known that u is continuous on Y, and that h^{-1} is continuous on X. Thus uh^{-1} is continuous on X. This implies that $e^{uh^{-1}}$ is an admissible exponential representation of $f(x)-p$ on X. The uniqueness of uh^{-1} up to an additive constant follows immediately from the fact that X is connected (for X is homeomorphic to a connected space). q.e.d.

B. Index of an interval or a simple arc. Consider now a continuous function f from a simple arc ab to Z. Let p denote a point of $Z-f(ab)$.

Definition B-1. Let $e^{u(x)}$ be an admissible exponential representation of $f(x)-p$ on ab. (It has been shown that there exists at least one such representation.) According to theorem A-3 and A-2, the difference $u(b)-u(a)$ is independent of the particular continuous branch u of the logarithm of $f(x)-p$ on ab which is used to compute it. Define

$$n_{ab}(f,p) = u(b)-u(a).$$

The value $\frac{1}{2\pi i} \cdot n_{ab}(f,p)$ is said to be the index of ab with respect to the point p under the function f.

Theorem B-1. The following three statements hold.

(i) $n_{ab}(f,p) = n_{ab}(f-p,0)$, where the function $f-p$ is defined by

$$(f-p)(x) = f(x)-p \text{ for } x \varepsilon ab.$$

(ii) If $f(x)-p = f_1(x) \cdot f_2(x)$ for $x \varepsilon ab$, where f_1 and f_2 are continuous functions from ab to Z, then

$$n_{ab}(f,p) = n_{ab}(f_1,0) + n_{ab}(f_2,0).$$

(iii) Define $h(z) = cz + d$ for $z \varepsilon Z$, where c and d are complex constants such that $c \neq 0$, then

$$n_{ab}(hf,h(p)) = n_{ab}(f,p).$$

Proof. Let a function g be defined as follows:

$$g(x) = f(x)-p \text{ for } x\varepsilon ab.$$

It is clear that $0\notin g(ab)$ and that g is continuous on ab. If $e^{u(x)}$ is an admissible exponential representation of $f(x)-p$ on ab, then

$$e^{u(x)} = f(x)-p = g(x) = g(x)-0 \text{ for } x\varepsilon ab.$$

It then follows that

$$n_{ab}(g,0) = u(b)-u(a) = n_{ab}(f,p).$$

It is not difficult to establish (ii), for it has been shown that there exist admissible exponential representations

$$e^{u(x)} = f(x)-p, \; e^{u_1(x)} = f_1(x), \text{ and } e^{u_2(x)} = f_2(x)$$

for $x\varepsilon ab$. It follows that

$$e^{u_1(x) + u_2(x)} = f_1(x) \cdot f_2(x) = f(x)-p = e^{u(x)} \text{ for } x\varepsilon ab.$$

This implies that u and $u_1 + u_2$ differ at most by an additive constant, and therefore

$$(u_1(b) + u_2(b)) - (u_1(a) + u_2(a)) = u(b)-u(a),$$

which means that

$$n_{ab}(f_1,0) + n_{ab}(f_2,0) = n_{ab}(f,p).$$

To prove (iii), observe that $hf(x) = h(p)$ is impossible for $x \varepsilon ab$, since h is one-to-one. Define a function k as follows:

$$k(x) = c \text{ for } x \varepsilon ab .$$

Thus, by (i), it is seen that

$$n_{ab}(hf,h(p)) = n_{ab}(k \circ (f-p),0),$$

where $f-p$ is defined as in (i). It follows from (ii) that

$$n_{ab}(hf,h(p)) = n_{ab}(k,0) + n_{ab}(f-p,0).$$

It is easily seen from the definition of k that $n_{ab}(k,0) = 0$. Then

$$n_{ab}(hf,h(p)) = n_{ab}(f-p,0).$$

This means, by (i), that

$$n_{ab}(hf,h(p)) = n_{ab}(f,p).$$

Statements (i), (ii), and (iii) have therefore been established. The proof of theorem B-1 is complete.

Theorem B-2. Suppose that $f(a) = f(b)$, then the following three statements hold.

(i) $\frac{1}{2\pi i} n_{ab}(f,p)$ is an integer. It is, if considered as a function of p on $Z-f(ab)$ for a fixed f, continuous on $Z-f(ab)$.

(ii) If p and q are points of the same component of the set $A-f(ab)$, then

$$n_{ab}(f,p) = n_{ab}(f,q).$$

(iii) If p is in the unbounded component of $Z-f(ab)$, then

$$n_{ab}(f,p) = 0 .$$

Proof. Let $q \in Z-f(ab)$, with $q \neq p$. Define a function g as follows:

$$g(x) = \frac{p-q}{f(x)-p} \quad \text{for } x \in ab.$$

(Note that $g(x) = 0$ is impossible for $x \in ab$.) Clearly, $g(x) \cdot f(x) - p \cdot g(x) = p-q$ for $x \in ab$. It follows that

$$(g(x) + 1) \cdot f(x) - p \cdot (g(x)+1) = f(x)-q \text{ for } x \in ab,$$

and therefore

$$(f(x)-p) \cdot (g(x) + 1) = f(x)-q \text{ for } x \in ab .$$

This shows in particular that $g(x) = -1$ is impossible for $x \varepsilon ab$. By (i) and (ii) of theorem B-1, it is seen that

$$n_{ab}(f,p) + n_{ab}(g,-1) = n_{ab}(f,q). \tag{I}$$

Let r be defined by

$$r = \text{dist } (p,f(ab)) = \sup \left\{|p-f(x)| : x \varepsilon ab\right\} .$$

It is immediately seen that r is a positive real number, since $f(ab)$ is a compact subset of Z. Suppose that q is restricted by the condition

$$|q-p| < r .$$

It follows that $g(ab) \subset D$, where D is defined by

$$D = \left\{z: z \varepsilon Z, |z| < 1\right\} .$$

The set $g(ab)$ cannot therefore intersect the ray from $(-1,0)$ to ∞ on the negative real axis. Let $e^{v(x)}$ be an admissible exponential representation of $g(x) + 1$ on ab, so that

$$e^{v(x)} = g(x) + 1 \text{ for } x \varepsilon ab .$$

Since $f(a) = f(b)$, it follows that $g(a) = g(b)$. Thus

$$e^{v(a)} = e^{v(b)} .$$

By (ii) of theorem A-1, it is true that

$$|\mathrm{Im}(v(a)) - \mathrm{Im}(v(b))| < 2\pi .$$

It follows that $v(a) = v(b)$, which implies that

$$n_{ab}(g, -1) = v(b) - v(a) = 0 .$$

Combining this equality with equality (I), it is seen that

$$n_{ab}(f, p) = n_{ab}(f, q)$$

whenever q is restricted by the condition $|q-p| < r$. Statement (i) will be proved if $n_{ab}(f, p)$ can be shown to be an integral multiple of $2\pi i$. To show that this is the case, it is only necessary to use the condition $f(a) = f(b)$, which is given in the hypothesis.

Statement (ii) follows immediately from statement (i) and from the fact that every component of $Z-f(ab)$ is a connected set.

To prove (iii), observe that there exists a point q in the unbounded component Q of $Z-f(ab)$ which is the origin of a ray lying completely in Q. It follows from (ii) that

$$n_{ab}(f,p) = n_{ab}(f,q) .$$

The fact that $n_{ab}(f,q) = 0$ follows from the equality $f(a) = f(b)$ and from (ii) of theorem A-1, by an argument similar to the one used above in verifying the equality $n_{ab}(g,-1) = 0$. This completes the proof of theorem B-2.

Theorem B-3. Let h be a sense-preserving homeomorphism from the simple arc ab to the simple arc cd ; then

$$n_{ab}(f,p) = n_{cd}(fh^{-1},p).$$

If h does not preserve sense, then

$$n_{ab}(f,p) = - n_{cd}(fh^{-1},p).$$

Proof. First suppose that h is sense-preserving. This means that $h(a) = c$ (and thus $h(b) = d$). Let g be defined to be the composite function fh^{-1}, whose domain is the arc cd. Let $e^{u(x)}$ be an admissible exponential representation of $f(x)-p$ on ab. Define v to be the composite function uh^{-1}; then

$$e^{v(x)} = e^{uh^{-1}(x)} = fh^{-1}(x)-p = g(x)-p \text{ for } x\varepsilon cd.$$

(The equality $g(x)-p = 0$ is clearly impossible for $x\varepsilon cd$.) Since v is continuous on cd, it follows that $e^{v(x)}$ is an admissible exponential representation of $g(x)-p$ on cd. Notice that

$$v(d)-v(c) = uh^{-1}(d)-uh^{-1}(c) = u(b)-u(a),$$

which means that

$$n_{ab}(f,p) = n_{cd}(fh^{-1},p),$$

since $g = fh^{-1}$.

Secondly suppose that h does not preserve sense. This means that $h(a) = d$ (and thus $h(b) = c$). The same argument as used above can be applied, obtaining

$$v(d)-v(c) = uh^{-1}(d)-uh^{-1}(c) = u(a)-u(b),$$

which implies that

$$n_{ab}(f,p) = - n_{cd}(fh^{-1},p) . \qquad \text{q.e.d.}$$

Theorem B-4. Let w be a complex-valued function defined and continuous on $h(ab)$, where h is a continuous function defined on ab. Let $p \in Z-w(h(ab))$. Suppose further that $h(a) = h(b)$. If $q \in h(ab) - \{h(a)\}$, then there exists a closed interval cd and a continuous function g on cd such that $g(c) = g(d) = q$, $g(cd) = h(ab)$, and $n_{ab}(wh, p) = n_{cd}(wg, p)$.

Proof. By theorems A-3 and B-3, it will suffice to prove this theorem for the case in which ab is a closed, real interval. Let $q \varepsilon h(ab) - \{h(a)\}$. Suppose that $c \varepsilon h^{-1}(\{q\})$, so that $c \varepsilon (a,b)$. Define $d = b-a+c$.

If $e^{u(x)}$ is an admissible exponential representation of $wh(x)-p$ on ab, define

$$g(x) = h(x) \text{ for } x \varepsilon [c,b],$$

$$g(x) = h(x-b+a) \text{ for } x \varepsilon (b,d],$$

$$v(x) = u(x) \text{ for } x \varepsilon [c,b], \text{ and}$$

$$v(x) = u(x-b+a) + u(b) - u(a) \text{ for } x \varepsilon (b,d] .$$

It is immediately seen that g and v are continuous on $[c,d]$, since $h(b) = h(a)$. Let $x \varepsilon [c,b]$; then

$$e^{v(x)} = e^{u(x)} = wh(x)-p = wg(x)-p.$$

Let $x \varepsilon (c,d]$; then

$$e^{v(x)} = e^{u(x-b+a)+u(b)-u(a)} = e^{u(x-b+a)} = wh(x-b+a)-p = wg(x)-p,$$

since, by (i) of theorem B-2, $e^{u(b)-u(a)} = 1$.

It has been shown that

$$e^{v(x)} = wg(x)-p \quad \text{for } x \varepsilon [c,d],$$

which means that $e^{v(x)}$ is an admissible exponential representation of $wg(x)-p$ on $[c,d]$.

To complete this proof, notice that

$$v(d)-v(c) = (u(d-b+a) + u(b)-u(a))-u(c) = u(b)-u(a),$$

since $d-b+a = c$. It follows from this equality that

$$n_{ab}(wh,p) = n_{cd}(wg,p). \qquad \text{q.e.d.}$$

Theorem B-5. Let $\{x_0, x_1, \ldots, x_n\}$ be a subdivision of ab, then

$$n_{ab}(f,p) = \sum_{i=o}^{n-1} n_{x_i x_{i+1}}(f,p) .$$

Proof. Let $e^{u(x)}$ be an admissible exponential representation of $f(x)-p$ on ab. It follows that $e^{u(x)}$ is an admissible exponential representation of $f(x)-p$ on $x_i x_{i+1}$ for each $i = 0, 1, \ldots, n-1$. Hence

$$n_{ab}(f,p) = u(b)-u(a) = \sum_{i=o}^{n-1} (u(x_{i+1})-u(x_i)) = \sum_{i=o}^{n-1} n_{x_i x_{i+1}}(f,p). \quad \text{q.e.d.}$$

Theorem B-6.

$$n_{ab}(f,p) = - n_{ba}(f,p).$$

Proof. If $e^{u(x)}$ is an admissible exponential representation of $f(x)-p$ on ab, then

$$n_{ab}(f,p) = u(b)-u(a) = -(u(a)-u(b)) = - n_{ba}(f,p). \qquad \text{q.e.d.}$$

C. Traversals of simple arcs and simple closed curves.

Definition C-1. Let T be a simple arc. A function h is said to be a traversal of T if and only if h is a homeomorphism from some closed, real interval $[a,b]$ onto T. Let J be a simple closed curve. A function h is said to be a traversal of J if and only if h is a continuous function from a closed, real interval $[a,b]$ onto J such that $h(a) = h(b)$ but h is one-to-one on (a,b).

Definition C-2. Let h and g be two traversals of a simple closed curve J. Therefore h and g are continuous functions from closed, real intervals $[a,b]$ and $[c,d]$ respectively onto J. The traversals h and g of J are said to agree in sense if and only if, for any three distinct points p, q, and r of J, the order of the permutation of the triplet (p,q,r) which $h(x)$ takes on as x moves from a to b is the same as the order of the permutation of the triplet (p,q,r) which $g(x)$ takes on as x moves from c to d.

Theorem C-1. Let J be a simple closed curve. Let h and g be two traversals of J agreeing in sense. Let f be a continuous function from J to Z. If $p \varepsilon Z - f(J)$, then

$$n_{ab}(fh, p) = n_{cd}(fg, p),$$

where $[a, b]$ and $[c, d]$ are the closed, real intervals associated with h and g respectively. If h and g do not agree in sense, then

$$n_{ab}(fh, p) = - n_{cd}(fg, p).$$

Proof. By theorem B-4, it is permissible to assume that $h(a) = g(c)$. Therefore $h(a) = h(b) = g(c) = g(d)$. Define a function w from $[a, b]$ to $[c, d]$ as follows:

$$w(x) = g^{-1}h(x) \quad \text{for} \quad x \varepsilon (a, b),$$

$$w(a) = c, \text{ and}$$

$$w(b) = d .$$

It is clear that w is a one-to-one, bicontinuous function from $[a, b]$ onto $[c, d]$. This means that w is a homeomorphism from $[a, b]$ onto $[c, d]$. Moreover, it is seen that w is sense-preserving. By theorem B-3, it is immediately seen that

$$n_{ab}(fh,p) = n_{cd}(fhw^{-1},p).$$

Clearly,

$$fhw^{-1} = fhh^{-1}g = fg \text{ on } (c,d). \text{ Also,}$$

$$fhw^{-1}(c) = fh(a) = fg(c) \quad (\text{since } h(a) = g(c)) \text{ and}$$

$$fhw^{-1}(d) = fh(b) = fg(d) \quad (\text{since } h(b) = g(d)).$$

It follows that

$$fhw^{-1} = fg \quad \text{on } [c,d] .$$

Thus $n_{ab}(fh,p) = n_{cd}(fg,p)$.

Now suppose that h and g do not agree in sense. The function w is now defined as follows:

$$w(d) = a,$$

$$w(c) = b, \quad \text{and}$$

$$w(x) = h^{-1}g(x) \quad \text{for} \quad x \varepsilon (c,d) .$$

It is easily seen that w is a homeomorphism from $[c,d]$ onto $[a,b]$, but w is not sense-preserving.

By theorem B-3, it follows that

$$n_{cd}(fg,p) = - n_{ab}(fgw^{-1},p).$$

It is clear that $fgw^{-1} = fgg^{-1}h = fh$ on (a,b). Also,

$$fgw^{-1}(a) = fg(d) = fh(a) \quad (\text{since } g(d) = h(a)), \text{ and}$$

$$fgw^{-1}(b) = fg(c) = fh(b) \quad (\text{since } g(c) = h(b)).$$

Thus $fgw^{-1} = fh$ on $[a,b]$. It follows that

$$n_{cd}(fg,p) = - n_{ab}(fh,p) \qquad \text{q.e.d.}$$

Definition C-3. Let J be a simple closed curve. Let f be a continuous function from J to Z. Suppose that $p \varepsilon Z - f(J)$. Then let A, B, and C be three distinct points of J and AB, BC, and CA denote the arcs of J containing just two of these points. By theorems B-3 and C-1, for any traversal h of J it is true that $n_{ab}(fh,p) = \pm [n_{AB}(f,p) + n_{BC}(f,p) + n_{CA}(f,p)]$, where $[a,b]$ is the closed, real interval associated. The value $n_{ab}(fh,p)$ will be written as $n_{ABCA}(f,p)$ if and only if the above equality holds with $+$ sign.

Theorem C-2. Define $f_1(x) = p + re^{2\pi ix}$ and $f_2(x) = p + re^{-2\pi ix}$ for $x \varepsilon [0,1]$, where p is a fixed complex number and r a positive real number. It is clear that both $f_1([0,1])$ and $f_2([0,1])$ are the circle with center p and radius r, and that f_1 and f_2 are two traversals of this circle not agreeing in sense. Then

$$n_{01}(f_1,p) = 2\pi i, \text{ and}$$

$$n_{01}(f_2,p) = - 2\pi i .$$

Proof. By hypothesis, $f_1(x)-p = re^{2\pi ix} = e^{\log r+2\pi ix}$ for $x\varepsilon[0,1]$. Hence

$$n_{01}(f_1,p) = (\log r + 2\pi i) - (\log r) = 2\pi i .$$

Similarly,

$$n_{01}(f_2,p) = - 2\pi i. \quad \text{q.e.d.}$$

Theorem C-3. Let $C = \{z: |z-p| = r\}$ be the circle with center p and radius r. Then all traversals of C have index +1 or -1 with respect to p.

Proof. Let $f_1(x) = p + re^{2\pi ix}$ and $f_2(x) = p + re^{-2\pi ix}$ for $x\varepsilon[0,1]$. It has been seen that f_1 and f_2 are two traversals of C not agreeing in sense.

If f is any traversal of C then f must agree in sense with f_1 or with f_2. By theorems C-1 and C-2, it follows that

$$n_{ab}(f,p) = \pm\ 2\pi i ,$$

where $[a,b]$ is the closed, real interval associated with f. q.e.d.

Theorem C-4. Let f be a traversal of a simple closed curve J in the complex plane Z. Let P be a point in the bounded component of $Z - J$, then

$$n_{ab}(f,p) = \pm\ 2\pi i,$$

where $[a,b]$ is the closed, real interval associated with f.

Proof. Let APB be the horizontal line segment containing P and having only its endpoints A and B in common with J. Let A be the right end of APB. Next, let C be a circle with center P lying entirely in the bounded component of $Z-J$. Let U and V be the respective points of AP and PB lying on C. Let S and T be interior points of the respective upper and lower semi-circles of C with ends U and V. Also, let X and Y be interior points on the arcs of J from A to B such that P is in the unbounded components of the complements of the simple closed curves

$$J_1 = AU + USV + VB + BXA, \text{ and}$$

$$J_2 = AU + UTV + VB + BYA.$$

Let h_1 and h_2 be traversals of J_1 and J_2, respectively. Let $[a_1, b_1]$ and $[a_2, b_2]$ be the closed, real intervals associated with h_1 and h_2 respectively. By (iii) of theorem B-2, it follows that

$$n_{a_1b_1}(h_1,P) = n_{a_2b_2}(h_2,P) = 0 .$$

By theorems B-2 and B-5, it is seen that

$$0 = n_{a_1b_1}(h_1,P) = n_{AU}(j,P) + n_{USV}(j,P) + n_{VB}(j,P) + n_{BXA}(j,P), \text{ and}$$

$$0 = n_{a_2b_2}(h_2,P) = n_{AU}(j,P) + n_{UTV}(j,P) + n_{VB}(j,P) + n_{BYA}(j,P) ,$$

where j is the identity function from J onto J.

Applying theorem B-6, it is seen that

$$n_{AXB}(j,P) = n_{AU}(j,P) = n_{USV}(j,P) + n_{VB}(j,P), \text{ and}$$

$$n_{BYA}(j,P) = n_{UA}(j,P) + n_{VTU}(j,P) + n_{BV}(j,P) .$$

Hence

$$n_{AXB}(j,P) + n_{BYA}(j,P) = n_{USV}(j,P) + n_{VTU}(j,P), \text{ or}$$

$$n_{AXBYA}(j,P) = n_{USVTU}(j,P) .$$

If it is supposed that h_1 and h_2 agree in sense with f in their common arcs (and this can of course be done), this last equation implies that

$$n_{ab}(f,P) = n_{USVTU}(j,P).$$

By theorem C-3,

$$n_{USVTU}(j,P) = \pm 2\pi i .$$

It then follows that $n_{ab}(f,P) = \pm 2\pi i$. q.e.d.

<u>Definition C-4.</u> Let J be a simple closed curve in the complex plane. A traversal f of J is said to be positive (negative) if and only if $\frac{1}{2\pi i}\, n_{ab}(f,p) = 1$ $\left(\frac{1}{2\pi i}\, n_{ab}(f,p) = -1\right)$, where p is any point of the bounded component of $Z-J$ and $[a,b]$ is the closed, real

interval associated with f.

Definition C-5. Let f be a continuous function from a simple closed curve J in the complex plane to Z. Let $p \varepsilon Z-f(J)$. The symbol $n_J(f,p)$ will denote the value $n_{ab}(fh,p)$, where h is a positive traversal of J and $[a,b]$ is the closed, real interval associated with h.

Theorem C-5. Let J be a simple closed curve in the complex plane. Let f and h be two traversals of J. Then f and h agree in sense if and only if they are both positive or both negative.

Proof. Let p denote a point of the bounded component of $Z-J$. Let $[a,b]$ and $[c,d]$ be the respective closed, real intervals associated with f and h.

First suppose that f and h agree in sense. It follows from theorem C-1 that $n_{ab}(f,p) = n_{cd}(h,p)$. According to theorem C-4, it follows that $n_{ab}(f,p) = n_{cd}(h,p) = \pm 2\pi i$. Thus f and h must be both positive or both negative.

Secondly suppose that f and h do not agree in sense. It follows from theorem C-1 that $n_{ab}(f,p) = - n_{cd}(h,p)$. By theorem C-4, it is true that $n_{ab}(f,p) = \pm 2\pi i$. It follows that one and only one of the traversals f and h can be positive, the other being negative. q.e.d.

Theorem C-6. Let J be a simple closed curve in the complex plane. Let f be a continuous function from J to Z. Let $p \varepsilon Z-f(J)$. Suppose that h_1 and h_2 are two positive traversals of J and also that g_1 and g_2 are two negative traversals of J, then

(i) $n_{ab}(fh_1,p) = n_{cd}(fh_2,p)$,

(ii) $n_{uv}(fg_1,p) = n_{xy}(fg_2,p)$, and

(iii) $n_{ab}(fh_1,p) = - n_{xy}(fg_1,p)$,

where $[a,b]$, $[c,d]$, $[u,v]$, and $[x,y]$ are the closed, real intervals associated with h_1, h_2, g_1, and g_2 respectively.

Proof. These are easy consequences of theorems C-1 and C-5.

Definition C-6. Let J and K be two simple closed curves in the complex plane. Let a, b, and c be three distinct points of J and let x,y, and z be three distinct points of K. Let p and q be points of the bounded components of $Z-J$ and $Z-K$ respectively. The senses (a,b,c) and (x,y,z) on J and K respectively are said to agree if and only if $n_{abca}(i,p) = n_{xyzx}(i,q)$, where i is the identity function. (Note that by therem C-5, this definition agrees with definition C-2 in the case where $J = K$.)

Theorem C-7. Let J and K be two simple closed curves in the complex plane and suppose that K lies in the bounded component of $Z-J$. Let a,b, and c be three distinct points of J, and let x,y, and z be three distinct points of K. Then the senses (a,b,c) and (x,y,z) on J and K respectively agree if and only if there exist disjoint simple arcs ax, by, and cz lying, except for their endpoints, in the annular region between J and K .

Proof. (a) The condition is sufficient. Suppose that the three arcs described in the condition exist. Let p be a point of the bounded component of $Z-K$ (and hence of the bounded component of $Z-J$). Then p is in the unbounded component of each of the complements of the simple closed curves

$$U = ab + by + yx + xa,$$

$$V = bc + cz + zy + yb, \text{ and}$$

$$W = ca + ax + xz + zc .$$

By theorem B-2, it follows that

$$n_{ab}(i,p) + n_{by}(i,p) + n_{yx}(i,p) + n_{xa}(i,p) = 0,$$

$$n_{bc}(i,p) + n_{cz}(i,p) + n_{zy}(i,p) + n_{yb}(i,p) = 0, \text{ and}$$

$$n_{ca}(i,p) + n_{ax}(i,p) + n_{xz}(i,p) + n_{zc}(i,p) = 0 ,$$

where i is the identity function.

From these three equalities it is seen that

$$(n_{ab}(i,p) + n_{bc}(i,p) + n_{ca}(i,p)) + (n_{xz}(i,p) + n_{zy}(i,p) + n_{yx}(i,p)) = 0 ,$$

which means that

$$n_{abca}(i,p) + n_{xzyx}(i,p) = 0 .$$

According to theorem C-6, this implies that

$$n_{abca}(i,p) = n_{xyzx}(i,p) .$$

This, by definition C-6, means that the senses (a,b,c) and (x,y,z) on J and K respectively agree.

(b) The condition is necessary. Suppose that the arcs, ax, by, and cz do not exist as asserted, but that the senses (a,b,c) and (x,y,z) on J and K respectively agree. Then there exist disjoint arcs ax and by lying, except for their endpoints, in the annular region between J and K. Now c can be joined to a point q in the set $K - \{x,y\}$ be a simple arc cq lying, except for its endpoints, in the annular region between J and K and not intersecting ax or by. Since z cannot be so joined, the point q must be on the arc xy of K and not on the arc xzy of K. It follows from the first part of the proof that the senses (a,b,c) on J and (x,y,q) on K must agree. This is obviously impossible since (x,y,z) and (x,y,q) are opposite senses on K, whereas they agree with (a,b,c) on J. q.e.d.

<u>Definition C-7</u>. Let J and K be two disjoint simple closed curves which constitute the boundary of a cylinder surface D (i.e. a set homeomorphic with a closed circular ring). Let a, b, and c be three distinct points of J, and let x,y, and z be three distinct points of K. The senses (a, b, c) on J and (x,y,z) on K are said to agree if and only if there exist three disjoint simple arcs ax, by, and cz lying, except for their endpoints, interior to the surface D .

Definition C-8. Let A and B be two 2-cells with edges J and K respectively. Suppose that A and B lie together in the interior of a 2-cell E with edge C. Two traversals f and h of J and K respectively are said to agree if and only if each of them agrees with one and the same traversal of C.

Theorem C-8. Let E be a 2-cell; then agreement in sense for two simple closed curves interior to E is invariant under any homeomorphism on E.

Proof. This follows immediately from definitions C-7 and C-8.

Theorem C-9. Let $J = xay + xby$ and $K = xay + xcy$ be two simple closed curves on a 2-cell having an arc xay in common, then the following two statements hold.

(i) If the interior of J contains the interior of K, then positive traversals of J and K agree in sense on xay.

(ii) If the interiors of J and K are disjoint, then positive traversals of J and K are opposite in sense on xay.

Proof. According to theorem C-8, it is permissible to assume that J and K lie in a plane.

Statement (i) is an immediate consequence of theorem C-7. A circle C can be taken interior to both J and K so that there will exist disjoint simple arcs xk, al, and ym in the annular region between K and C, with k, l, and m on C. Thus if $yaxby$ is the sense in which J is described by a positive traversal of J, then $yaxcy$ must be the sense in which K is described by a positive traversal of K, for the sense (y, a, x) agrees in both cases with the sense (k,l,m) on C (by theorem C-5). Positive traversals of J

and K therefore agree in sense on xay.

To prove statement (ii), it may be assumed that $yaxby$ is the sense in which J is described by a positive traversal of J. Then $xbycx$ is the sense in which L is described by the positive traversals of L, where $L = xby + xcy$. It follows that $xaycx$ must be the sense in which K is described by a positive traversal of K, by the first part of this proof. Since xay and yax are opposite, statement (ii) is established.

Definition C-9. Let h be a homeomorphism from a plane X onto a plane Y. Then h is said to be positive if and only if, for any positive traversal f of any simple closed curve in C in X, hf is a positive traversal of the simple closed curve $h(C)$ in Y.

Theorem C-10. Let X be the complex plane (i.e. $X = Z$). Define a function h as follows:

$$h(x) = cx + d \quad \text{for } x \in X ,$$

where c and d are complex constants such that $c \neq 0$. Then h is a positive homeomorphism from X onto Z.

Proof. It is obvious that h is a homeomorphism from X onto Z.

Now suppose that C is any simple closed curve in X. Let f be a positive traversal of C. The set $h(C)$ is clearly a simple closed curve in Z.

By theorem B-1, it follows that

$$n_{ab}(hf, h(p)) = n_{ab}(f, p) = 2\pi i$$

for all points p in the bounded component of $X-C$, where $[a,b]$ is the closed, real interval associated with f. Thus hf is a positive traversal of $h(C)$. q.e.d.

Theorem C-11. Let h be a positive homeomorphism from a plane X onto a plane Y. Let g be a positive homeomorphism from Y onto a plane W. Then gh is a positive homeomorphism from X onto W.

Proof. Let f be a positive traversal of a simple closed curve C in X. By hypothesis, hf is a positive traversal of the simple closed curve $h(C)$ in Y. By hypothesis, g is positive, thus ghf is a positive traversal of the simple closed curve $gh(C)$ in W. q.e.d.

D. Index invariance.

Theorem D-1. Let f and g be continuous functions from a simple arc ab into Z. If $|g(x)-f(x)| < |f(x)-p|$ for all $x \varepsilon ab$ and $f(a) = g(a)$ and $f(b) = g(b)$, then

$$n_{ab}(f,p) = n_{ab}(g,p) .$$

Proof. Let $e^{u(x)}$ be an admissible exponential representation of $f(x)-p$ on ab. Thus

$$u(x) = \log |f(x)-p| + iU(x) \text{ for } x \varepsilon ab ,$$

where U is real-valued and continuous on ab.

For every x in ab, the circle with center at $f(x)-p$ and radius $|g(x)-f(x)|$ neither contains nor encloses the origin, and $g(x)-p$ is on such a circle, since $|(g(x)-p)-(f(x)-p)|=|g(x)-f(x)|$.

Then there exists a unique, real-valued, continuous function V on ab such that

$$e^{\log|g(x)-p| + iV(x)} = g(x)-p \quad \text{for } x\epsilon ab, \text{ and}$$

$$U(x) - \frac{\pi}{2} < V(x) < U(x) + \frac{\pi}{2} \quad \text{for } x\epsilon ab \text{ .}$$

Set $v(x) = \log|g(x)-p| + iV(x)$ for $x\epsilon ab$, so that v is continuous on ab. It is clear that $e^{v(x)}$ is an admissible exponential representation of $g(x)-p$ on ab.

Now observe that

$$u(b) - v(b) = i(U(b) - V(b)) \text{ since } f(b) = g(b), \text{ and}$$

$$u(a) - v(a) = i(U(a) - V(a)) \text{ since } f(a) = g(a) \text{ .}$$

It follows that

$$\text{(i)} \quad n_{ab}(f,p) - n_{ab}(g,p) = i[(U(b) - V(b)) - (U(a) - V(a))], \text{ and}$$

therefore

$$\text{(ii)} \quad n_{ab}(f,p) - n_{ab}(g,p) = 2\pi\, i(m-q) \text{ ,}$$

where m and q are integers. Since $|U(x) - V(x)| < \frac{\pi}{2}$ for $x\epsilon ab$, it is immediately seen from (i) that

$$\left|n_{ab}(f,p) - n_{ab}(g,p)\right| < \pi .$$

Then (ii) can hold only if $m = q$, so that

$$n_{ab}(f,p) = n_{ab}(g,p). \qquad \text{q.e.d.}$$

Theorem D-2. The conclusion of theorem D-1 holds if the conditions $f(a) = g(a)$ and $f(b) = g(b)$ are replaced by the conditions $f(a) = f(b)$ and $g(a) = g(b)$.

Proof. Repeat the proof of theorem D-1 through the definition of the function v. If the notations of theorem D-1 are kept, it follows that

$$u(b) - v(b) = (\log|f(b) - p| + iU(b)) - (\log|g(b) - p| + iV(b)), \text{and}$$

$$u(a) - v(a) = (\log|f(a) - p| + iU(a)) - (\log|g(a) - p| + iV(a)).$$

The hypothesis that $f(b) = f(a)$ and $g(b) = g(a)$ imply that

$$(u(b) - v(b)) - (u(a) - v(a)) = i[(U(b) - V(b)) - (U(a)-V(a))],$$

which means that

$$n_{ab}(f,p) - n_{ab}(g,p) = 2\pi i(m-q),$$

where m and q are integers. Then the argument follows in the same manner as in theorem D-1.

Theorem D-3. Let J be a simple closed curve. Let f and g be continuous functions from J to Z. If $|g(z)-f(z)|<|f(z)-p|$

for $z \varepsilon J$, then $n_{ab}(fh,p) = n_{ab}(gh,p)$ for any traversal h of J, where $[a,b]$ is the closed, real interval associated with h.

Proof. This is obvious. Apply theorem D-2 to the functions fh and gh.

Theorem D-4. Let f be a continuous function from a separable metric space X into Z such that $f(X) \neq Z$, and let $p \varepsilon Z - f(X)$. Let ab and cd be simple arcs in X. Suppose that there exists a sense-preserving homeomorphism h from ab onto cd such that

(i) $|f(x) - fh(x)| < |f(x) - p|$ for $x \varepsilon ab$, and

(ii) $f(a) = fh(a)$ and $f(b) = fh(b)$.

Then $n_{ab}(f,p) = n_{cd}(f,p)$.

Proof. It follows from theorem D-1 that

$$n_{ab}(f,p) = n_{ab}(fh,p) .$$

By theorem B-3, it is true that

$$n_{ab}(fh,p) = n_{cd}(f,p) .$$

From these two equalities it follows that

$$n_{ab}(f,p) = n_{cd}(f,p) . \qquad \text{q.e.d.}$$

E. Traversals of region boundaries and region subdivisions.

Definition E-1. A bounded plane region R is said to be elementary if and only if $Fr(R)$ consists of a finite number of simple closed curves $J_0,\ldots,J_n$ such that J_0 encloses the set $J_1+\ldots+J_n$. The curve J_0 is said to be the outer boundary of R. By a positive traversal of $Fr(R)$ is meant an $(n+1)$-tuple $h=(h_0,\ldots,h_n)$, where h_0 is a positive traversal of J_0 and h_i is a negative traversal of J_i for each $i=1,\ldots,n$.

Now let f be a continuous function from $Fr(R)$ to Z. Let p be any point of $Z-f(Fr(R))$. If $h=(h_0,\ldots,h_n)$ and $g=(g_0,\ldots,g_n)$ are two positive traversals of $Fr(R)$, it follows from theorems C-1 and C-5 that

$$\sum_{i=o}^{n} n_{a_i b_i}(fh_i,p) = \sum_{i=o}^{n} n_{c_i d_i}(fg_i,p),$$

where $[a_i,b_i]$ and $[c_i,d_i]$ are the respective closed, real intervals associated with h_i and g_i for each $i=0,\ldots,n$. The sum $\sum_{i=o}^{n} n_{a_i b_i}(fh_i,p)$ thus defines a complex number which is independent of h. This complex number will be denoted by $n_{Fr(R)}(f,p)$. The value $\frac{1}{2\pi i}\, n_{Fr(R)}(f,p)$ is said to be the index of $Fr(R)$ with respect to p under the function f.

<u>Definition E-2</u>. A subset G of Z is said to be a graph if and only if G is the sum of a finite set V of points of Z and a finite number of open arcs such that their endpoints are distinct and belong to V. The points of V are said to be the vertices of G and the arcs described above are said to be the edges of G.

Definition E-3. Let R be an elementary region. Let $Fr(R) = \sum_{i=o}^{n} Ji$, where J_o encloses the set $J_1+\ldots+J_n$. Let G be a graph such that $Fr(R) \subset G \subset \overline{R}$. Then G subdivides R into a finite number of regions $R_1,\ldots,R_m$ each bounded by a simple closed curve C_i of G. The intersection of two of the C_i's is either empty or a vertex of G or an edge of G. Let $h = (h_1,\ldots,h_m)$ be an m-tuple of positive traversals of the simple closed curves $C_1,\ldots,C_m$, respectively. Each edge which lies interior to R lies on exactly two of the curves $C_1,\ldots,C_m$ and is therefore traversed once positively and once negatively by h. Any edge of G lying on J_o lies on exactly one of the curves $C_1,\ldots,C_m$ and is therefore traversed once positively by h. Any edge of G lying on one of $J_1,\ldots,J_m$ is similarly traversed once by h, but negatively. Thus h induces a positive traversal of $Fr(R)$. It is said that h is a positive traversal of G.

Theorem E-1. With the same assumptions and notations of the preceding definition, let f be a continuous function from G to Z. Let $p \varepsilon Z-f(G)$. Then $n_{Fr(R)}(f,p) = \sum_{i=1}^{m} n_{a_i b_i}(fh_i,p) = \sum_{i=1}^{m} n_{Fr(R_i)}(f,p)$, where $[a_i,b_i]$ is the closed, real interval associated with h_i, for each $i=1,\ldots,m$.

Proof. By definition, $n_{Fr(R)}(f,p) = \sum_{j=o}^{n} n_{c_j d_j}(fg_j,p)$, where $g=(g_1,\ldots,g_n)$ is the positive traversal of $Fr(R)$ induced by h, $[c_j,d_j]$

being the closed, real interval associated with g_j for each $j=0,\ldots,n$. From the preceding definition and theorem B-5 it is immediately seen that

$$n_{Fr(R)}(f,p) = \sum_{j=o}^{n} n_{c_j d_j}(fg_j,p) = \sum_{i=1}^{m} n_{a_i b_i}(fh_i,p). \qquad \text{q.e.d.}$$

Theorem E-2. Let f be a continuous function from $\overline{R}$ to Z, where R is an elementary region with $Fr(R) = \sum_{i=o}^{n} J_i$ (J_o enclosing the set $J_1+\ldots+J_n$). If $p\varepsilon Z-f(\overline{R})$, then $n_{Fr(R)}(f,p) = 0$.

Proof. Let G be a graph such that $Fr(R) \subset G \subset \overline{R}$. This graph G can be taken in such a way that the subdivision of R induced by G has mesh so small that

$$\text{diam } f(\overline{R}_i) < \text{dist } (p,f(\overline{R}))$$

for every region R_i into which R is divided by G. The proof of this can be found in G. T. Whyburn [1].

It follows that p is in the unbounded component of the complement of each such $f(\overline{R}_i)$. By theorems E-1 and B-2, it is seen that

$$n_{Fr(R)}(f,p) = \sum_{i=1}^{m} n_{Fr(R_i)}(f,p) = 0,$$

where $R_1,\ldots,R_m$ are the subregions of R induced by G. q.e.d.

F. The index of an analytic function. The purpose of this section is to show that, for analytic functions, the given definition of the topological index agrees with the classical one, which uses the concept of contour integral.

Theorem F-1. Let f be a complex function analytic on a region R of the complex plane Z. Let C be a continuously differentiable closed curve in R. If $p \varepsilon Z - f(C)$, then

$$n_C(f,p) = \int_C \frac{f'(z)}{f(z) - p}\, dz.$$

Proof. Let h be any positive traversal of C. Let $[a,b]$ be the closed, real interval associated with h. The composite function fh is obviously continuous on $[a,b]$. According to theorem A-2, $fh(x) - p$ has an admissible exponential representation $e^{u(x)}$ on $[a,b]$. Then

$$e^{u(x)} = fh(x) - p \quad \text{for } x \varepsilon [a,b]. \qquad \text{(i)}$$

By hypothesis, h' exists and is continuous on $[a,b]$. Since f is analytic on R, it follows that the function g, defined by

$$g(x) = fh(x) - p \quad \text{for } x \varepsilon [a,b],$$

has a derivative at every point of $[a,b]$. Moreover,

$$g'(x) = f'(h(x)).h'(x) \quad \text{for } x \varepsilon [a,b]. \qquad \text{(ii)}$$

It is easy to show that (i) and the existence of g' on $[a,b]$ imply the existence of u' on $[a,b]$. By (i), it now follows that

$$e^{u(x)}.u'(x) = g'(x) = f'(h(x)).h'(x) \quad \text{for } x \varepsilon [a,b]. \qquad \text{(iii)}$$

Thus, $u'(x) = e^{-u(x)}.f'(h(x)).h'(x)$ for $x \varepsilon [a,b]$, which shows that u' is continuous on $[a,b]$.

The definition of contour integral implies that

$$\int_C \frac{f'(z)}{f(z)-p}\,dz = \int_a^b \frac{f'(h(t))\cdot h'(t)}{f(h(t))-p}\,dt .$$

By (i) and (iii), it follows that

$$\int_C \frac{f'(z)}{f(z)-p}\,dz = \int_a^b \frac{e^{u(t)}\cdot u'(t)}{e^{u(t)}}\,dt = \int_a^b u'(t)\,dt = u(b)-u(a).$$

(The integral $\int_a^b u'(t)\,dt$ obviously exists, for u' has been shown to be continuous on $[a,b]$.) By the definition of $n_C(f,p)$, this last equality means that

$$\int_C \frac{f'(z)}{f(z)-p}\,dz = n_C(f,p) . \qquad \text{q.e.d.}$$

<u>Remark</u>. Note that this proof applies, with minor modifications, to the case where C is a piecewise continuously differentiable closed curve in R.

<u>G. Examples</u>. The purpose of this section is to illustrate some of the theorems of this chapter and to illustrate how to compute the index in special cases.

<u>Example G-1</u>. This example refers to theorem C-1. Let J be the unit circle in the complex plane. Let h and g be the traversals of J defined by

$$h(x) = e^{\pi ix} \text{ for } x \varepsilon [-1,1], \text{ and}$$

$$g(x) = e^{2\pi ix} \text{ for } x \varepsilon [0,1] .$$

Let $f(z) = z^n$ for $z \varepsilon Z$, where n is a fixed positive integer. It is desired to verify that $n_{01}(fg,0) = n_{-11}(fh,0)$.

Observe that

$$fh(x)-0 = (h(x))^n = e^{n\pi ix} \text{ for } x\varepsilon[-1,1], \text{ and}$$

$$fg(x)-0 = (g(x))^n = e^{2n\pi ix} \text{ for } x\varepsilon[0,1] .$$

It follows from the definition of the index that

$$n_{-11}(fh,0) = (n\pi i) - (-n\pi i) = 2n\pi i, \text{ and}$$

$$n_{01}(fg,0) = (2n\pi i) - (0) = 2n\pi i .$$

Thus $n_{-11}(fh,0) = n_{01}(fg,0)$.

Example G-2. This example refers to theorem D-1. Let p be an arbitrary complex number, and define

$$f(x) = p + re^{2\pi ix} \text{ for } x\varepsilon[0,1], \text{ and}$$

$$g(x) = p + \tfrac{3}{4}\, re^{2\pi ix} \text{ for } x\varepsilon[0,1] ,$$

where r is a positive real number. It is obvious that $p\varepsilon Z-f([0,1])$ and $p\varepsilon Z-g([0,1])$. Observe now that

$$|f(x)-p| = |re^{2\pi ix}| = r \text{ for } x\varepsilon[0,1], \text{ and}$$

$$|f(x)-g(x)| = |re^{2\pi ix}(1-\tfrac{3}{4})| = \tfrac{1}{4}r \text{ for } x\varepsilon[0,1] \ .$$

Thus $|f(x)-g(x)| < |f(x)-p|$ for $x\varepsilon[0,1]$. It is desired to verify that

$$n_{01}(f,p) = n_{01}(g,p) \ .$$

It is immediately seen that

$$f(x)-p = e^{\log r + 2\pi ix} \text{ for } x\varepsilon[0,1], \text{ and}$$

$$g(x)-p = e^{\log \frac{3}{4}r + 2\pi ix} \text{ for } x\varepsilon[0,1] \ .$$

By definition, it follows that

$$n_{01}(f,p) = (\log r + 2\pi i) - (\log r) = 2\pi i, \text{ and}$$

$$n_{01}(g,p) = (\log \tfrac{3}{4}r + 2\pi i) - (\log \tfrac{3}{4}r) = 2\pi i \ .$$

Thus $n_{01}(f,p) = n_{01}(g,p)$.

CHAPTER II

APPLICATIONS TO COMPLEX VARIABLES

Only complex functions will be considered in this chapter. Thus the expression "f is a function on a set X" means "f is a function from a set X to the complex plane." The complex plane will be denoted by Z.

A. Properties of the index of continuous functions.

Theorem A-1. Let f and g be continuous functions on $\overline{R}$, where R is an elementary region. Suppose that

$$|g(z)| < |f(z)| \text{ for } z \varepsilon \mathrm{Fr}(R) ,$$

then $n_{\mathrm{Fr}(R)}(f+g,0) = n_{\mathrm{Fr}(R)}(f,0)$.

Proof. First notice that $f(z) \neq 0$ for $z \varepsilon \mathrm{Fr}(R)$, and also that $f(z) + g(z) \neq 0$ for $z \varepsilon \mathrm{Fr}(R)$.

Let $z \varepsilon \mathrm{Fr}(R)$; then

$$f(z) + g(z) = f(z) \cdot (1 + \frac{g(z)}{f(z)}) .$$

This implies that

$$1 + \frac{g(z)}{f(z)} \neq 0 \text{ for } z \varepsilon \mathrm{Fr}(R) .$$

Let the function h be defined by

$$h(z) = 1 + \frac{g(z)}{f(z)} \text{ for } z \varepsilon \mathrm{Fr}(R) .$$

The function h is therefore continuous and does not vanish on its domain. By theorem B-1 of Chapter I, it follows that

$$n_{Fr(R)}(f+g,0) = n_{Fr(R)}(f,0) + n_{Fr(R)}(h,0).$$

The proof will be complete if $n_{Fr(R)}(h,0)$ can be shown to be equal to zero. Let $z \varepsilon Fr(R)$, then

$$|h(z)-1| = \left|\frac{g(z)}{f(z)}\right| < 1 ,$$

which means that

$$h(Fr(R)) \subset \left\{ z : |z-1| < 1 \right\} .$$

By theorem B-2 of Chapter I, it follows that

$$n_{Fr(R)}(h,0) = 0 ,$$

for the ray from 0 to ∞ on the negative real axis does not intersect $h(Fr(R))$. q.e.d.

<u>Theorem A-2</u>. Let f be a function continuous on a region R of the complex plane. Suppose that $f'(z_o)$ exists and is not zero for some point z_o of R . If C is a sufficiently small simple closed curve enclosing z_o, then

$$n_C(f,f(z_o)) = 2\pi i .$$

<u>Proof</u>. It is possible to write

$$f(z) - f(z_o) = (z-z_o) \cdot (f'(z_o) + e(z)) \text{ for } z \varepsilon R,$$

where $\lim_{z=z_0} e(z) = 0$.

The hypothesis that $f'(z_0) \neq 0$ implies that, for all sufficiently small simple closed curve C enclosing z_0, it is true that

(i) $Q + C \subset R$, where Q is the bounded component of $Z-C$,

(ii) $f(z) - f(z_0) \neq 0$ for $z \epsilon Q + C - \{z_0\}$, and

(iii) $|e(z)| < \frac{1}{2} |f'(z_0)|$ for $z \epsilon Q + C$.

Let h be a positive traversal of C. By theorem B-1 of Chapter I, it follows that

$$n_{ab}(fh,f(z_0)) = n_{ab}(h,z_0) + n_{ab}(eh, -f'(z_0)).$$

where $[a,b]$ is the closed, real interval associated with h . Since h is a positive traversal of C, it follows that

$$n_{ab}(h,z_0) = 2\pi i .$$

By (iii) and theorem B-2 of Chapter I, it is seen that

$$n_{ab}(eh, -f'(z_0)) = 0 .$$

It then follows that

$$n_C(f,f(z_0)) = n_{ab}(fh,f(z_0)) = 2\pi i. \quad \text{q.e.d.}$$

B. <u>Properties of the index for analytic functions.</u>

<u>Theorem B-1.</u> Let f be a non-constant function analytic on a simply connected region R of the complex plane. The set of zeros

of f cannot be uncountable since f is non-constant. Let z_j denote the sequence of zeros of f in R, each counted as many times as its order indicates. If C is a continuously differentiable closed curve in R which does not pass through any of the zeros of f in R, then

$$n_C(f,0) = \sum_j n_C(I,z_j),$$

where I denotes the identity function on C.

Proof. Only a finite number of zeros of f can be enclosed by C, for otherwise f would be identically zero on R. Suppose then that C encloses $z_1,\ldots,z_m$. Let G be the bounded component of $Z-C$.

It is known (see Ahlfors [3]) that f can be represented as follows:

$$f(z) = (z-z_1)(z-z_2)\ldots(z-z_m)g(z) \text{ for } z\varepsilon G,$$

where g is analytic and never vanishes on G. If $z\varepsilon G-\{z_1,\ldots,z_m\}$, then

$$\frac{f'(z)}{f(z)} = \frac{1}{z-z_1} +\ldots+\frac{1}{z-z_m} + \frac{g'(z)}{g(z)} .$$

By theorem F-1 of Chapter I, it follows that

$$n_C(f,0) = \int_C \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{m} \int_C \frac{1}{z-z_j}\,dz + \int_C \frac{g'(z)}{g(z)}\,dz .$$

By the Cauchy's integral theorem, it is seen that

$$\int_C \frac{g'(z)}{g(z)}\,dz = 0 .$$

It then follows that

$$n_C(f,0) = \sum_{j=1}^{m} \int_C \frac{1}{z-z_j}\,dz = \sum_{j=1}^{m} n_C(I,z_j) .$$

By theorem B-2 of Chapter I, it is true that

$$n_C(I,z_j) = 0 \text{ for } j=m+1,\ m+2\ ,\dots .$$

This implies that

$$n_C(f,0) = \sum_j n(I,z_j),$$

which is the conclusion of the theorem.

Theorem B-2. Let f be a non-constant function analytic on a simply connected region R of the complex plane. Let $\{p_j\}$ be the sequence of p-points of f in R, each counted as many times as its order indicates. If C is a continuously differentiable closed curve in R which does not pass through any of the p-points of f in R, then

$$n_C(f,p) = \sum_j n_C(I,p_j) ,$$

where I denotes the identity function on C .

Proof. According to theorem B-1 of Chapter I, it is true that

$$n_C(f,p) = n_C(f-p,0).$$

Observe now that $\{p_j\}$ is the sequence of zeros of $f-p$ in R. Theorem B-1 can therefore be applied to the function $f-p$, obtaining

$$n_C(f,p) = \sum_j n_C(I,p_j),$$

which means that

$$n_C(f,p) = \sum_j n_C(I,p_j). \quad \text{q.e.d.}$$

Theorem B-3. Let f be a non-constant function analytic on a simply connected region R of the complex plane. Let C be a continuously differentiable simple closed curve in R. Suppose that $p \varepsilon Z - f(C)$. Then the value $\frac{1}{2\pi i} n_C(f,p)$ is equal to the number of p-points of f enclosed by C, each counted as many times as its order indicates.

Proof. Let $\{p_j\}$ denote the sequence of p-points of f in R. Only a finite number of these points can be enclosed by C, for f is non-constant. It may be assumed that C encloses $p_1,\ldots,p_m$, where m is some fixed integer. By theorem B-2, it follows that

$$n_C(f,p) = \sum_j n_C(I,p_j),$$

where I is the identity function on C. The fact that C is simple and closed implies that

$$n_C(I,p_j) = 2\pi i \text{ for } j=1,\ldots,m.$$

By theorem B-2 of Chapter I, it is seen that

$$n_C(I,p_j) = 0 \text{ for } j=m+1,\ m+2, \ldots .$$

It is therefore true that

$$n_C(f,p) = (2\pi i) \cdot m,$$

which yields

$$\frac{1}{2\pi i}\, n_C(f,p) = m. \qquad \text{q.e.d.}$$

Theorem B-4. Let f be a non-constant function analytic on a simply connected region R of the complex plane. Let C be a continuously differentiable simple closed curve in R. Let p and q be points of the same component of $Z-f(C)$. The number of p-points of f enclosed by C is then equal to the number of q-points of f enclosed by C, if each p-point of f and each q-point of f are counted as many times as their orders indicate.

Proof. By theorem B-2 of Chapter I, it is seen that

$$n_C(f,p) = n_C(f,q).$$

This, by theorem B-4, implies the conclusion of the theorem.

Theorem B-5. Let f be a non-constant function analytic at a point z_o of Z, and set $w_o = f(z_o)$. Suppose that $(f-w_o)$ has a zero of order n at z_o. If e is a sufficiently small positive number, then there exists a corresponding positive number d such that, for all complex numbers w for which $|w-w_o| < d$, the equation

$f(z) = w$ has exactly n roots in the disk $\{z: |z-z_o| < e\}$.

Proof. It is known that e can be chosen so that

(i) f is analytic on $\{z: |z-z_o| < e\}$, and

(ii) z_o is the only zero of $(f-w_o)$ on $\{z: |z-z_o| \leq e\}$.

Let $C = \{z: |z-z_o| = e\}$. The way in which e was chosen implies that

$$w_o \in Z - f(C).$$

Then there exists a disk $D = \{w: |w-w_o| < d\}$ such that $D.f(C) = \emptyset$, for $f(C)$ is compact and $w_o \in Z-f(C)$. Thus $w \in D$ implies that w and w_o are in the same component of $Z-f(C)$. By hypothesis, z_o is a w_o-point of f of order n. This implies, by theorem B-4, the conclusion of the theorem.

Theorem B-6. A non-constant analytic function f maps open sets of Z onto open sets of Z.

Proof. It will suffice to prove that f maps interior points onto interior points. This follows immediately from theorem B-5, which says that the image under f of a sufficiently small disk $\{z: |z-z_o| < e\}$ contains a neighborhood of w_o, where z_o is a point of analyticity of f and $w_o = f(z_o)$.

C. Some properties of analytic functions which follow from the properties of the topological index.

Theorem C-1. (Rouché's theorem). Let f and g be two functions analytic on $\overline{R}$, where R is a region of the complex plane such that $Fr(R)$ is a continuously differentiable simple closed curve C. Suppose that

$$z \varepsilon C \text{ implies } |g(z)| < |f(z)|,$$

then f and $f+g$ have the same number of zeros in R, each zero being counted as many times as its order indicates.

Proof. This follows immediately from theorems A-1 and B-3.

Theorem C-2 (maximum modulus principle). Let f be a non-constant function analytic on a bounded open set G of the complex plane. Suppose that f is continuous on $Fr(G)$ and that $|f(z)| \leq M$ for $z \varepsilon Fr(G)$, then $|f(z)| < M$ for $z \varepsilon G$.

Proof. Suppose that $|f(z_0)| \geq M$ for some point z_0 of G. Let V be a neighborhood of z_0 such that $V \subset G$. By theorem B-6, $f(V)$ is open. It follows that there exists a point z_1 of V such that

(i) $|f(z_1)| > |f(z_0)| \geq M$.

It is known that

(ii) $|f(w)| = \max \left\{ |f(z)| : z \varepsilon \overline{G} \right\}$

for some point w of $\overline{G}$. By hypothesis, $|f(z)| \leq M$ for $z \varepsilon Fr(G)$. This and (i) imply that w is not a point of $Fr(G)$, and therefore $w \varepsilon G$. Since G is open, there exists a neighborhood U of w such that $U \subset G$. By theorem B-6, $f(U)$ is open. Then there exists a point z_2 of U such that

$$|f(z_2)| > |f(w)| ,$$

which is a contradiction to (ii). q.e.d.

Theorem C-3. Let f be a non-constant function analytic on a bounded open set G of the complex plane such that $G.L = \emptyset$, where

$L = \left\{ z: |f(z)| = k \right\}$ for some positive number k. If f is continuous on $Fr(G)$ and if $Fr(G) \subset L$, then $f(z_0) = 0$ for at least one point z_0 of G.

Proof. By theorem C-2, it is true that

(i) $|f(z)| < k$ for $z \varepsilon G$.

It is known that

$$|f(w)| = \min \left\{ |f(z)|: z \varepsilon \overline{G} \right\}$$

for some point w of $\overline{G}$. By (i), it follows that $w \varepsilon G$. Suppose that $f(w) \neq 0$, then $|f(w)| > 0$. Since $w \varepsilon G$, there exists a neighborhood V of w such that $V \subset G$. By theorem B-6, $f(V)$ is open. Then there exists a point z_1 of V such that $|f(z_1)| < |f(w)|$, which contradicts the definition of w. If the assumption that $f(w) \neq 0$ implies a contradiction, it must then be true that $f(w) = 0$. q.e.d.

Theorem C-4 (fundamental theorem of algebra). Every non-constant polynomial P has at least one zero.

Proof. Let $P(z_0) = w_0 \neq 0$ for some point z_0 of Z. Define $G = \left\{ z: |P(z)| < |w_0| \right\}$. Since the set $\left\{ w: |w| < |w_0| \right\}$ is open and P is continuous, it follows that $G = P^{-1} \left(\left\{ w: |w| < |w_0| \right\} \right)$ is open. Since $P(z) \to \infty$ as $z \to \infty$, it follows that G is bounded. The set G is therefore open and bounded. The conclusion of the theorem follows immediately from the fact that

$$Fr(G) \subset \left\{ z: |P(z)| = |w_0| \right\}$$

and from theorem C-3.

CHAPTER III

STABILITY

A. The concept of stability.

Definition A-1. Let f be a continuous function from a topological space X to a metric space Y with metric d. A point p of $f(X)$ is said to be an unstable value of f if and only if, for every positive real number b, there exists a continuous function h from X to Y such that

(i) $d(f(x),h(x)) < b$ for $x \varepsilon X$, and

(ii) $h(X) \subset Y - \{p\}$.

A point q of $f(X)$ is said to be a stable value of f if and only if q is not an unstable value of f.

Example A-1. Let R denote the real line. Let f be a continuous function from R into $[0, +\infty)$ such that $f(0) = 0$ and $\lim_{x \to +\infty} f(x) = +\infty$. Then zero is an unstable value of f, and p is a stable value of f for all positive real numbers p.

Proof. It is easy to see that zero is an unstable value of f.

Let p be any positive real number. Set $b = \frac{p}{2} > 0$. Let g be a continuous function from R into $[0,+\infty)$ such that $|f(x)-g(x)| < b$ for $x \varepsilon R$. It follows that $f(x) - b < g(x) < f(x) + b$ for $x \varepsilon R$. By hypothesis, this implies that

$$- b < g(o) < b \quad .$$

This shows that $o \leq g(o) < \frac{p}{2}$, for g is non-negative. Since $\lim_{x=+\infty} f(x) = +\infty$, a positive real number x can be chosen in such a way that $f(x) - b > 4b$. Thus

$$p < 2p = 4b < f(x) - b < g(x) .$$

Then $g(o) < \frac{p}{2} < p$ and $g(x) > 2p > p$. Since R is connected and g is continuous, it follows that there exists a real number y such that $o < y < x$ and $g(y) = p$. This means that $p \varepsilon g(x)$. Then p is a stable value of f. q.e.d.

<u>Theorem A-1</u>. Let f be a continuous function from a topological space X to a metric space Y with metric d. Let $A \subset X$ and define h to be the restriction to A of f. If a point p of $h(A)$ is a stable value of h, then p is a stable value of f.

<u>Proof</u>. Suppose that p is an unstable value of f. Thus, for every positive real number b, there exists a continuous function g_b from X to Y such that

(i) $d(g_b(x), f(x)) < b$ for $x \varepsilon X$, and

(ii) $p \varepsilon Y - g_b(X)$.

It then follows that

(iii) $d(g_b(x), h(x)) < b$ for $x \varepsilon A$, and

(iv) $p \varepsilon Y - g_b(A)$,

since $g_b(A) \subset g_b(X)$. This means, by definition, that p is an unstable value of h. q.e.d.

B. Topological equivalence of functions

Definition B-1. Let X_1, X_2, Y_1, and Y_2 be topological spaces. Suppose that f_i is a function from X_i to Y_i for each $i=1,2$. The function f_1 is said to be equivalent to the function f_2 if and only if there exist a homeomorphism x from X_1 onto X_2 and a homeomorphism y from Y_1 onto Y_2 such that

$$f_1(z) = y^{-1}f_2x(z) \quad \text{for } z \in X_1 .$$

Theorem B-1. Equivalence of functions is an equivalence relation.

Proof. (a) Let f be a function from a topological space X to a topological space Y. It is clear that f is equivalent to f, for if x is the identity function on X and if y is the identity function on Y, then

$$f(z) = y^{-1}fx(z) \text{ for } z \in X.$$

(b) Let f_i be a function from a topological space X_i to a topological space Y_i for each $i=1,2$. Suppose that f_1 is equivalent to f_2. Then there exist a homeomorphism x from X_1 onto X_2 and a homeomorphism y from Y_1 onto Y_2 such that

$$f_1(z) = y^{-1}f_2x(z) \quad \text{for } z \in X_1 .$$

It is now asserted that

$$f_2(z) = yf_1x^{-1}(z) \quad \text{for } z \in X_2,$$

which will prove that f_2 is equivalent to f_1. Let $z \in X_2$, then

$x^{-1}(z)\varepsilon X_1$.

By hypothesis,

$$f_1(x^{-1}(z)) = y^{-1}f_2x(x^{-1}(z)), \text{ or}$$

$$f_1x^{-1}(z) = y^{-1}f_2(z) .$$

Thus $f_2(z) = yf_1x^{-1}(z)$, since y is one-to-one.

(c) Let f_i be a function from a topological space X_i to a topological space Y_i for each $i=1,2,3$. Suppose that f_1 is equivalent to f_2 and that f_2 is equivalent to f_3. Then there exist homeomorphisms

x_1 from X_1 onto X_2,

x_2 from X_2 onto X_3,

y_1 from Y_1 onto Y_2, and

y_2 from Y_2 onto Y_3,

such that

$$f_1(z) = y_1^{-1}f_2x_1(z) \text{ for } z\varepsilon X_1, \text{ and}$$

$$f_2(z) = y_2^{-1}f_3x_2(z) \text{ for } z\varepsilon X_2 .$$

Let $z \varepsilon X_1$, then

$$f_1(z) = y_1^{-1} f_2 x_1(z) = y_1^{-1} y_2^{-1} f_3 x_2 x_1(z), \text{ or}$$

$$f_1(z) = (y_2\, y_1)^{-1}\, f_3(x_2\, x_1)\ (z),$$

which shows that f_1 is equivalent to f_3, for $y_2 y_1$ is a homeomorphism from Y_1 onto Y_3, and $x_2 x_1$ is a homeomorphism from X_1 onto X_3. q.e.d.

Theorem B-2. Let f_i be a continuous function from a topological space X_i to a metric space Y_i with metric d_i for each $i=1,2$. Suppose that f_1 is equivalent to f_2. Let x and y be the homeomorphisms described in the preceding definition. If y and y^{-1} are uniformly continuous, then a point z of $f_1(X_1)$ is a stable value of f_1 if and only if $y(z)$ is a stable value of f_2. (Notice that $y(z) \varepsilon f_2(X_2)$.)

Proof. First suppose that $y(z)$ is a stable value of f_2. Then there exists a positive real number b such that, for any function g from X_2 to Y_2 satisfying the condition $d_2(f_2(w), g(w)) < b$ for $w \varepsilon X_2$, it is true that $y(z) \varepsilon g(X_2)$.

By hypothesis, y is uniformly continuous. Then there exists a positive real number c such that

(i) $u \varepsilon Y_1$, $v \varepsilon Y_1$, and $d_1(u,v) < c$ imply $d_2(y(u), y(v)) < b$.

Let s be a function from X_1 to Y_1 such that

(ii) $d_1(s(w), f_1(w)) < c$ for $w \varepsilon X_1$.

Define t to be the composite function ysx^{-1}. Clearly t is a function from X_2 to Y_2. Let $w \varepsilon X_2$; then it is seen from (i) and (ii) that

$$d_2(t(w),\ f_2(w)) = d_2(ysx^{-1}(w), yf_1x^{-1}(w)) < b\ .$$

It follows from the definition of b that $y(z) \varepsilon ysx^{-1}(X_2)$. Since y is one-to-one, it is true that $z \varepsilon sx^{-1}(X_2)$. This implies that there exists a point p in X_2 such that $z = sx^{-1}(p)$.

Define $q = x^{-1}(p)$; then $q \varepsilon X_1$ and $z = s(q)$. It follows that $z \varepsilon s(X_1)$, which shows that z is a stable value of f_1.

The converse (i.e., that if z is a stable value of f_1, then $y(z)$ is a stable value of f_2) follows at once from theorem B-1. q.e.d.

C. Case of a continuous complex function.

Theorem C-1. Let R be an elementary region (in the complex plane Z) with $Fr(R) = J_0 + \ldots + J_n$, where J_0 encloses the set $J_1 + \ldots + J_n$. Let f be a continuous function from $\overline{R}$ to Z such that $f(\overline{R}) \neq Z$. Let $p \varepsilon Z - f(Fr(R))$. If $n_{Fr(R)}(f,p) \neq 0$ then p is a stable value of f.

Proof. By theorem E-2 of Chapter I, $n_{Fr(R)}(f,p) \neq 0$ implies that $p \varepsilon f(\overline{R})$. Let $L = f(Fr(R))$; then L is compact. Since $p \not\varepsilon L$, it follows that

$$d = dist(p,L) = \sup \left\{ |p - f(z)| : z \varepsilon Fr(R) \right\} > 0.$$

Then $|f(z) - p| \geq d$ for $z \varepsilon Fr(R)$.

Define $b = \frac{d}{2}$, then $b > 0$. Let h be a continuous function from $\overline{R}$ to Z such that

$$|f(z)-h(z)| < b \text{ for } z \varepsilon \overline{R}.$$

It is seen that

$$|f(z)-h(z)| < b < d \leq |f(z)-p| \text{ for } z \varepsilon Fr(R).$$

Applying theorem D-3 to each of $J_o, \ldots, J_n$, it is concluded that

$$n_{Fr(R)}(f,p) = n_{Fr(R)}(h,p) .$$

Thus $n_{Fr(R)}(h,p) \neq 0$, which implies that $p \varepsilon h(\overline{R})$. It follows that p is stable value of f . q.e.d.

Theorem C-2. Let f be a homeomorphism from an open set U of the complex plane onto an open set W of the complex plane. If $p \varepsilon W$, then p is a stable value of f .

Proof. Let C be a simple closed curve in U enclosing the point $f^{-1}(p)$ of U and such that the bounded component of $Z-C$ lies in U . Then $f(C)$ is a simple closed curve in W enclosing the point p of W and such that the bounded component of $Z-f(C)$ lies in W . It is known that

$$n_{f(C)}(I,p) = 2\pi i ,$$

where I denotes the identity function on $f(C)$. Denote by R the bounded component of $Z-C$. It is seen that R is an elementary region in U such that $Fr(R) = C$. Thus

$$n_{Fr(R)}(f,p) = n_C(f,p) = \pm\, n_{f(C)}(I,p) = \pm\, 2\pi i \neq 0.$$

By theorem C-1, it follows that p is a stable value of f . q.e.d.

Theorem C-3. Let R be a region of the complex plane. Let f be a non-constant function analytic on R. If $w_0 \varepsilon f(R)$, then w_0 is a stable value of f.

Proof. Let z_0 be a point of R such that $f(z_0) = w_0$. There exists a positive real number r such that

(i) $D + C \subset R$, and

(ii) the equation $f(z) = w$ has no solution for $z \varepsilon (D + C) - \{z_0\}$, where

$$D = \{z : |z-z_0| < r\} \text{ and } C = \{z : |z-z_0| = r\} .$$

It is clear that $C = Fr(D)$, and that D is an elementary region. By (ii), it is true that $w_0 \varepsilon Z - f(Fr(D))$. By theorem B-3 of Chapter 2, it follows that

$$n_C(f, w_0) = (2\pi i)m \neq 0,$$

where m is the order of the w_0-point z_0 of f. By theorem C-1, it is seen that w_0 is a stable value of f. q.e.d.

CHAPTER IV

SECOND-ORDER AUTONOMOUS SYSTEMS OF DIFFERENTIAL EQUATIONS

This chapter will be concerned with the system

$$S: \quad \left\{ \begin{array}{l} x'(t) = P(x(t),\ y(t)) \\ y'(t) = Q(x(t),\ y(t)) \\ -\infty < t < +\infty \end{array} \right\}$$

Such a system, in which the independent variable t does not appear explicitly on the right-hand side, is said to be autonomous. The real-valued functions P and Q are defined on some region D of the complex plane.

A. Basic definitions

Definition A-1. A point (x_o, y_o) of the complex plane is said to be a singular point of the system S if and only if $P(x_o, y_o) = 0$ and $Q(x_o, y_o) = 0$.

Definition A-2. A plane curve C is said to be a characteristic of the system S if and only if $C = \left\{ (x(t),\ y(t)):\ t \varepsilon (-\infty, +\infty) \right\}$, where (x, y) is a non-constant solution of the system S.

Definition A-3. Let (x_o, y_o) be an isolated singular point of the system S. The point (x_o, y_o) is said to be a stable singular point of the system S if and only if, for every positive number e, there exists a positive number d with the following property. If $C = \left\{ (x(t),\ y(t)):\ t \varepsilon (-\infty, +\infty) \right\}$ is a characteristic of the system S satisfying

$$[(x(t_o) - x_o)^2 + (y(t_o) - y_o)^2]^{\frac{1}{2}} < d$$

for some real number t_o, then

$$[(x(t) - x_o)^2 + (y(t) - y_o)^2]^{\frac{1}{2}} < e \text{ for } t \epsilon [t_o, +\infty).$$

Definition A-4. A singular point (x_o, y_o) of S is said to be simple if and only if there exist four real constants a, b, c, and d satisfying the condition $ad - bc \neq 0$ and such that if the functions e_1 and e_2 are defined by

$$P(x,y) = a(x-x_o) + b(y-y_o) + e_1(x,y), \text{ and}$$

$$Q(x,y) = c(x-x_o) + d(y-y_o) + e_2(x,y)$$

for $(x,y) \epsilon D$, then

$$\lim_{(x,y) \to (x_o,y_o)} \frac{e_i(x,y)}{[(x-x_o)^2 + (y-y_o)^2]^{\frac{1}{2}}} = 0 \text{ for } i = 1,2.$$

It can be shown that simple singularities are isolated.

Definition A-5. Let (x_o, y_o) be a simple singularity of the system S. Let a, b, c, and d be the four real constants described in definition A-4. Let l_1 and l_2 be the characteristic values of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then (x_o, y_o) is said to be a

(i) nodal point if and only if l_1 and l_2 are real and of the same sign

(ii) saddle point if and only if l_1 and l_2 are real and of opposite sign

(iii) vortex point if and only if l_1 and l_2 are purely imaginary and

(iv) spiral point if and only if l_1 and l_2 are complex but not purely imaginary.

Note. The preceding definition is not standard. The names nodal points, saddle point, etc., are motivated by the graphs of the characteristics of the system under consideration. The standard definition, however, agrees with the one given here when the system under consideration is S. The reader is referred to Coddington and Levinson [5] for geometric definitions of these concepts.

Theorem A-1. Let (x_o,y_o) be a simple singularity of the system S . The following statements hold.

(i) If (x_o,y_o) is a vortex point or a spiral point, then (x_o,y_o) is a stable singular point of S .

(ii) If (x_o,y_o) is a saddle point, then (x_o,y_o) is an unstable singular point of S .

(iii) If (x_o,y_o) is a nodal point, and if l_1 and l_2 are the characteristic values of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then (x_o,y_o) is a stable (unstable) singular point of S if l_1 and l_2 are both negative (positive).

Proof. The proof of this theorem will not be given here. The reader is referred to W. Hurewicz [4].

B. The index of a simple singularity of the system S.

It will be assumed throughout this section that (0,0) is a simple singular point of the system S . The complex function P + iQ will be denoted by F . Since $(0,0)$ is an isolated singularity of S, there exists a circle C with center at $(0,0)$ such that no other singularity of S is enclosed by C or contained in C. This notation shall be kept.

Definition B-1. The index of the simple singular point $(0,0)$ of S is defined to be the value $\frac{1}{2\pi i} n_C(f,(0,0))$.

Theorem B-1. The simple singular point $(0,0)$ of S is a saddle point if and only if its index is -1.

Proof. The proof of this theorem will not be given here. The reader is referred to W. Hurewicz [4] .

Example B-1. Consider the system

$$S_1: \quad \left\{ \begin{array}{l} x'(t) = -x(t) \\ y'(t) = -y(t) \\ -\infty < t < +\infty \end{array} \right\} .$$

Here $P(x,y) = -x$ and $Q(x,y) = -y$ for $(x,y)\varepsilon Z$, where Z denotes the complex plane. It is seen that $(0,0)$ is a simple singular point of the system S_1 . In this case the function F is defined by

$$F(x,y) = P(x,y) + iQ(x,y) = -x - iy = -(x+iy) \quad \text{for } (x,y)\varepsilon Z .$$

In other words, $F(z) = -z$ for $z\varepsilon Z$. By theorem A-2 of chapter II, it is seen that $\frac{1}{2\pi i} n_C(F, (0,0)) = 1$, for F is analytic at $z=(0,0)$ and $F'(z)\neq 0$.

By theorem B-1, it follows that $(0,0)$ is not a saddle point.

To check this result, it is only necessary to find the characteristic values of the matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. They are found to be

$l_1 = l_2 = -1$. Thus $(0,0)$ is a nodal point.

Example B-2. Consider the system

$$S_2: \left\{ \begin{array}{l} x'(t) = y(t) \\ y'(t) = x(t) \\ -\infty < t < +\infty \end{array} \right\}$$

Here $P(x,y) = y$ and $Q(x,y) = x$ for $(x,y)\varepsilon Z$, where Z denotes the complex plane. It is seen that $(0,0)$ is a simple singular point of the system S_2. The function F is defined by

$$F(x,y) = y + ix = i(x-iy) \quad \text{for } (x,y)\varepsilon Z .$$

In other words, $F(z) = i\bar{z}$ for $z\varepsilon Z$. The function F is not analytic at any point of Z, but the value $n_C(F,(0,0))$ can be found easily. Let

$$h(t) = re^{2\pi it} \quad \text{for} \quad t\varepsilon[0,1]$$

be a positive traversal of C, where r is the radius of C. It is then seen that

$$Fh(t) = ire^{-2\pi it} \text{ for } t\varepsilon[0,1] .$$

Thus $n_C(F,(0,0)) = n_{01}(Fh,(0,0)) = (-2\pi i) - 0 = -2\pi i$.

The index of $(0,0)$ is therefore -1. By theorem B-1, it follows that $(0,0)$ is a saddle point.

To check this result, it is seen that the characteristic values of the matrix $\binom{0\ 1}{1\ 0}$ are $l_1 = 1$ and $l_2 = -1$. Thus $(0,0)$ is a saddle point.

<u>Example B-3</u>. Consider the system

$$S: \quad \left\{ \begin{array}{l} x'(t) = P(x(t),\ y(t)) \\ y'(t) = Q(x(t),\ y(t)) \\ -\infty < t < +\infty \end{array} \right\}$$

where the function $F = P + iQ$ is analytic on the complex plane Z. It is assumed that F is non-constant, that $F(0) = 0$, and that $F'(0) \neq 0$. Under these assumptions it is asserted that $(0,0)$ is a simple singular point of S, and that it is not a saddle point.

To prove this, observe that (by Taylor's theorem) it is possible to write

$$P(x,y)=P(x,y)-P(0,0)=xD_1P(0,0)+yD_2P(0,0)+\tfrac{1}{2}[x^2D_{1,1}P(u,v)+2xyD_{1,2}P(u,v)+y^2D_{2,2}P(u,v)]$$

where (u,v) is a point in the open line segment joining $(0,0)$ and (x,y). Define $a = D_1P(0,0)$, $b = D_2P(0,0)$, and $e_1(x,y) = \frac{1}{2}[x^2D_{1,1}P(u,v) + 2xy\ D_{1,2}P(u,v) + y^2D_{2,2}P(u,v)]$. It then follows that

$$P(x,y) = ax + by + e_1(x,y) .$$

In a similar way it can be seen that

$$Q(x,y) = cs + dy + e_2(x,y),$$

where $c = D_1Q(0,0)$, $d = D_2Q(0,0)$, and

$$e_2(x,y) = \tfrac{1}{2}[x^2D_{1,1}Q(w,z) + 2xy\ D_{1,2}Q(w,z) + y^2D_{2,2}Q(w,z)] ,$$

where (w, z) is a point in the open line segment joining $(0,0)$ and (x,y).

Observe that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} D_1P(0,0) & D_2P(0,0) \\ D_1Q(0,0) & D_2Q(0,0) \end{vmatrix} = |F'(0)|^2 \neq 0$$

using the Cauchy-Riemann equations.

It remains to be shown that $\lim\limits_{(x,y)=(0,0)} \dfrac{e_i(x,y)}{(x^2 + y^2)^{\frac{1}{2}}} = 0$

for $i = 1, 2$. This will be proved here for e_1, for the proof for e_2 is analogous. By changing to polar coordinates, it is seen that

$$e_1(r \cos p, r \sin p) = \frac{r^2}{2}[\cos^2 p\, D_{1,1}P(u,v) + 2 \sin p \cos p D_{1,2}P(u,v) + \sin^2 p D_{2,2}P(u,v)]$$

where $x = r \cos p$ and $y = r \sin p$.

The expression in brackets is seen to remain bounded as r approaches zero and a bound can be found independent of the value of p, for P and Q have continuous partial derivatives of every order. It follows that

$$\lim_{r=0} \frac{e_1(r \cos p, r \sin p)}{r} = 0$$

for any value of p, which is equivalent to the desired result. The point $(0,0)$ is therefore a simple singular point of S. By theorem A-2 of Chapter II, it is seen that $\frac{1}{2\pi i} n_c(F,(0,0)) = 1$. By theorem B-1, it follows that $(0,0)$ is not a saddle point. q.e.d.

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